A Dirac Morphism for the Farrell-Jones Isomorphism Conjecture in K-Theory

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Abstract

We construct a Dirac Morphism. We prove that if this Dirac morphism is invertible, then the isomorphism conjecture for non-connective algebraic K-theory holds true.

It is known, (see [MV] and [MN)that the Baum-Connes isomorphism conjecture on a given group G follows from the existence of an invertible Dirac morphism associated to that group. In [MN] the authors prove the existence of a Dirac morphism which suffices to construct the assembly map in the setting of the Baum-Connes conjecture. Here, given a group G we also construct an associated Dirac morphism, and we show that if it is invertible, then the Farrell-jones isomorphism conjecture holds true for that group by using theorem 1 stated below.

DEFINITION 1:(see [MH], [N] and [AH]) *Given a triangulated category $\mathcal{T}$, we say that $\mathcal{G}$ is a set of weak generators of this category if this set is closed under (de-) suspensions and if we get that $X = 0$, for X an object of $\mathcal{T}$ if for all G in $\mathcal{G}$ and all integer n*

$$Hom_{\mathcal{T}}(\Sigma^n G, X) = 0$$

DEFINITION 2: (see [MH], [SS] and [HPS]) *A stable model category is a pointed stable model category for which the functors Σ and Ω on the homotopy category are invertible.*

REMARK 1: (see [MH], [SS] and [HPS]) The homotopy category of a stable model category is a triangular category which has all limits and colimits, infinite coproducts and products.

DEFINITION 2:(see [S]) *Let $\mathcal{T}$ be a triangulated category with infinite coproducts. An object C in $\mathcal{T}$ is called compact if for any family $(Y_i)_{i\in I}$ of objects of $\mathcal{T}$ one has a natural isomorphism*

$$\sqcup Hom_{\mathcal{T}}(C, Y_i) \mapsto Hom_{\mathcal{T}}(C, \sqcup Y_i)$$

Equivalently any morphism $C \mapsto \sqcup Y_i$ factorizes through a finite subcoproduct.

DEFINITION 3: (see [N]) *Let $\mathcal{T}$ be a triangulated category. A full additive subcategory $\mathcal{S}$ is called a triangular subcategory if every object isomorphic to an object of $\mathcal{S}$ is in $\mathcal{S}$, if $\mathcal{S}$ is closed under suspensions and if for any distinguished triangle*

$$X \mapsto Y \mapsto Z \mapsto \Sigma X$$

such that the objects X and Y are in $\mathcal{S}$, the object Z is also in $\mathcal{S}$.

DEFINITION 4: (see [S]) *A triangular subcategory $\mathcal{S}$ of a triangular category $\mathcal{T}$ is called thick if it is closed under direct summands.*

DEFINITION 5: (see [S]and [SS])*A triangular subcategory $\mathcal{S}$ of a triangular category $\mathcal{T}$ is called localizing if it is closed under taking arbitrary coproducts.*

REMARK 2: By definition, if $\mathcal{S}$ is localizing then it is thick.

DEFINITION 6: (see [SS]) *Given a triangular category $\mathcal{T}$ with infinite coproducts, a set $\mathcal{G}$ of objects of $\mathcal{T}$ is called a generating set of $\mathcal{T}$ if the smallest localizing subcategory of $\mathcal{T}$ containing $\mathcal{G}$ is the whole category $\mathcal{T}$.*

LEMMA 1: (see [SS] lemma 2.2.1) *Let $\mathcal{T}$ be a triangular category with infinite coproducts and let $\mathcal{G}$ be a set of compact objects in $\mathcal{T}$. Then $\mathcal{G}$ is a set of generators of $\mathcal{T}$ if and only if $\mathcal{G}$ is a set of weak generators of $\mathcal{T}$.*

REMARK 3: (see [R]) By remark 1, the homotopy category $Ho(\mathcal{K})$ of a stable model category $\mathcal{K}$ is a triangular category with infinite coproducts, and

moreover a set of objects $\mathcal{G}$ in this homotopy category is a weak generating set if and only if it is a generating set.

DEFINITION 7: (see [AH] and [N]) i) *An object T in a triangular category $\mathcal{T}$ with infinite coproducts is called λ-small for a regular cardinal λ, if any map $T \mapsto \sqcup X_i$ into an arbitrary coproduct in $\mathcal{T}$ factors through some subcoproduct*

$$T \mapsto \sqcup X_j \mapsto \sqcup X_i$$

with J a subset of I such that $Card(J)$ is less than λ.

ii) *A set $\mathcal{G}$ of objects of $\mathcal{T}$ is called λ-perfect if it satisfies:*

a) $0 \in \mathcal{G}$

b) *Any map $G \mapsto \sqcup T_i$ with G in $\mathcal{G}$, $(T_i)_{i \in I}$ in $\mathcal{T}$ and $card(I)$ less than λ factors as*

$$G \mapsto \sqcup G_i \mapsto \sqcup f_i : \sqcup G_i \mapsto \sqcup T_i$$

with G_i in $\mathcal{G}$ and the maps $f_i : G_i \mapsto T_i$ in $\mathcal{T}$.

iii) *A set $\mathcal{G}$ in $\mathcal{T}$ is called λ-compact for a regular cardinal λ if every G in $\mathcal{G}$ is λ-small, and also $\mathcal{G}$ is λ-perfect.*

REMARK 4: Any compact object of a triangular category is a λ-compact object for every regular cardinal λ.

DEFINITION 8:(see [N],[S] and [AH]) *A triangular category $\mathcal{T}$ is called well generated if it has a weak generating set $\mathcal{G}$ of λ-compact objects, for some regular cardinal λ.*

REMARK 5: In particular, by remark 4, a triangular category $\mathcal{T}$ with infinite coproducts and a generating set of compact objects, or equivalently, a weak generating set of compact objects, that is, a compactly generated triangular category, is a well generated category.

PROPOSITION 1: ([K1]) *A triangular category with infinite coproducts is well generated provided that there is a weak generating set $\mathcal{G}$ consisting of λ-*

small objects such that for any family of maps $X_i \mapsto Y_i$ with $i \in I$ and with induced maps

$$Hom_{\mathcal{T}}(G, X_i) \mapsto Hom_{\mathcal{T}}(G, Y_i)$$

being surjective for all G in $\mathcal{G}$, the induced map

$$Hom_{\mathcal{T}}(G, \sqcup X_i) \mapsto Hom_{\mathcal{T}}(G, \sqcup Y_i)$$

is also surjective.

REMARK 6: The Brown Representability theorem holds for cohomology functors defined over well generated triangular categories (see [N]), but in [K2] it is proven that the Brown Representability theorem holds for cohomology functors under the weaker hypothesis which is the one of proposition 1 but this time deleting the λ-small condition.

PROPOSITION 2:(see [AH])*Let $\mathcal{T}$ be a well generated triangular category and let $\mathcal{L}$ be a localizing subcategory which is generated by a set of objects. Then $\mathcal{L}$ is also well generated. Moreover the Verdier quotient $\mathcal{T}/\mathcal{L}$ is a localization of $\mathcal{T}$ and is well generated.*

PROPOSITION 3: (see [R]) *Let $\mathcal{K}$ be a pointed combinatorial model category, then its homotopy category $Ho(\mathcal{K})$ is well generated.In particular the homotopy category of a stable combinatorial category is well generated.*

PROPOSITION 4:(see [CGR] Th 3.9)*Let $\mathcal{K}$ be a stable combinatorial model category. If Vopenska's principle holds then every localizing subcategory of $Ho(\mathcal{K})$ is single generated.*

See [AR] for information on the large-cardinal axiom of set theory called Vopenska's principle.

REMARK 7: It then follows by propositions 2,3 and 4 that under Vopenska's principle, every localizing subcategory of the homotopy category of a stable combinatorial model category is well generated and hence we can apply Brown's representability theorem to that subcategory.

We are going to apply the definitions, lemmas and propositions stated above

to a particular model category, which is $Spt^{Or(G)}$ See [Hir] or [BM1] for the definition of the model structure on $Spt^{Or(G)}$. Here Spt is the model category of spectra of compactly generated Hausdorff spaces with the stable model category structure. $Or(G)$ is the orbit category of a group G. The objects are the homogeneous spaces G/H with H a subgroup of G considered as left G-sets, and the morphisms are G-maps $G/H \mapsto G/K$ given by right multiplication $r_g : G/H \mapsto G/K, g_1H \mapsto g_1gK$ provided $g \in G$ satisfies $g^{-1}Hg \subset K$.

In particular we need to apply proposition 3 to $Spt^{Or(G)}$, hence we must prove that this category is combinatorial and stable. It is stable since Spt is stable. Recall also that a combinatorial category is a model category which is cofibrantly generated (see [Hir], and is locally presentable, that is, it is cocomplete and accesible, (see [AR]). Since it is a model category, by definition it is cocomplete, so that we must only see that it is accesible and cofibrantly generated. Now Spt is accesible, moreover it is combinatorial (see [R] ex 3.5 iv))and therefore also $Spt^{Or(G)}$ is accesible by ([AR] th2.39, page 96).Also by [BM1 see Th3.5] $Spt^{Or(G)}$ is cofibrantly generated, and hence $Spt^{Or(G)}$ is a combinatorial stable model category and it follows by proposition 3 that the homotopy category of $Spt^{Or(G)}$ is a well generated triangular category and by remark 7 every localizing subcategory of the homotopy category of $Spt^{Or(G)}$ is also well generated so that we can apply Brown's Representability Theorem to it.

In [BM2] a Quillen adjunction $ind_{\mathcal{D}}^{\mathcal{C}} : Spt^{\mathcal{D}} \Leftrightarrow Spt^{\mathcal{C}} : res_{\mathcal{D}}^{\mathcal{C}}$ is defined where $\mathcal{C}$ is the orbit category $Or(G)$ and where $\mathcal{D}$ stands for the subcategory $Or(G, (\mathcal{VC})$ whose objects G/H are such that H is a virtually cyclic subgroup. There is an induced derived adjunction $Lind_{\mathcal{D}}^{\mathcal{C}} : Ho(Spt^{\mathcal{D}}) \Leftrightarrow Ho(Spt^{\mathcal{C}}) : res_{\mathcal{D}}^{\mathcal{C}}$.

In [LR], page 797, given an associative ring R with unit, a suitable functor $K_R : Or(G) \mapsto Spt$ is constructed. This functor has the property that the homotopy groups $\pi_*(K_R(G/H))$ are canonically isomorphic to the nonconnective algebraic K theory groups $K_*(R[H])$ for all subgroup H of G, where $R[H]$ is the group ring defined by the group H.

The following theorem is proven in [BM2]

THEOREM 1: The Farrell-Jones conjecture for nonconnective K-theory is verified for a group G if and only if the image of the functors K_R in $Ho(Spt^{\mathcal{C}})$, for all associative rings with unit R belong to $Lind_{\mathcal{D}}^{\mathcal{C}}(Ho(Spt^{\mathcal{D}}))$

It is known, (see [MV] and [MN)that the Baum-Connes isomorphism conjecture on a given group G follows from the existence of an invertible Dirac morphism associated to that group. In [MN] the authors prove the existence of a Dirac morphism which suffices to construct the assembly map in the setting of

the Baum-Connes conjecture. Here, given a group G we also construct an associated Dirac morphism, and we show that if it is invertible, then the Farrell-jones isomorphism conjecture holds true for that group by using theorem 1.

DEFINITION 9: *We define by* $(\mathcal{CI})$ *for the localizing subcategory generated by* $Lind^{\mathcal{C}}_{\mathcal{D}}(Ho(Spt^{\mathcal{D}}))$. $(\mathcal{CI})$ *is a localizing subcategory of the triangular category* $Ho(Spt^{\mathcal{C}})$

By analogy with the definition 4.5 in [MN] we define

DEFINITION 10: Given X, an object of $Ho(Spt^{\mathcal{C}})$, a $(\mathcal{CI})$ approximation of X is a morphism $f : \widehat{X} \mapsto X$ with $\widehat{X}$ belonging to $(\mathcal{CI})$ such that $Lres^{\mathcal{C}}_{\mathcal{D}}(f)$ is invertible.

DEFINITION 11 (Dirac morphism): Let $*_{Or(G)}$ be a unit in the symmetric monoidal category $Ho(Spt^{\mathcal{C}})$ and use the notation $*_{Or(G)}$ or $*_{\mathcal{C}}$ for the sake of simplicity, then a Dirac morphism is a $(\mathcal{CI})$ approximation of $*_{\mathcal{C}}$.

By [CGR] under Vopenska's principle, every localizing subcategory of the homotopy category of a stable combinatorial model category is coreflective. A full subcategory $\mathcal{C}$ of a category $\mathcal{T}$ is called coreflective if the inclusion $\mathcal{C} \hookrightarrow \mathcal{T}$ has a right adjoint.The composite $C : \mathcal{T} \mapsto \mathcal{T}$ is called colocalization onto $\mathcal{C}$. $\mathcal{C}$ is then the class of C-colocal objects. Dually, a reflection $\mathcal{L}$ of $\mathcal{T}$ is a full subcategory such that the inclusion $\mathcal{L} \hookrightarrow \mathcal{T}$ has a left adjoint $\mathcal{T} \mapsto \mathcal{L}$. Then $L : \mathcal{T} \mapsto \mathcal{T}$ is called localization onto $\mathcal{L}$. $\mathcal{L}$ is then the class of L-local objects. By theorem 1.4 in [CGR] there is a bijective correspondence between coreflections and reflections, and given a coreflection $\mathcal{C}$ with colocalization $C : \mathcal{T} \mapsto \mathcal{T}$ there is a triangle

$$CX \mapsto X \mapsto LX \mapsto \Sigma(CX)$$

for all X in $\mathcal{T}$, where $L : \mathcal{T} \mapsto \mathcal{T}$ is the localization associated to the reflection $\mathcal{L}$ which corresponds to $\mathcal{C}$. Here $\mathcal{L}$ coincides with the set of all X in $\mathcal{T}$ such that $Hom_{\mathcal{T}}(\Sigma^k Y, X) = 0$ for all Y in $\mathcal{C}$ and all integer k.

Observe that our localizing category $(\mathcal{CI})$ is by the results of [CGR] coreflective, therefore for each object X in $Ho(Spt^{Or(G)})$ there is a triangle

$$P \mapsto X \mapsto N \mapsto \Sigma P$$

with P belonging to $(\mathcal{CI})$ and N belonging to an orthogonal reflective category (orthogonal in the sense stated above) noted $(\mathcal{CC})$.This an analogous result to the one in [M] Th 70.

Next we will show that the map $P \mapsto X$ from above is a $(\mathcal{CI})$ approximation of X. Note that in that case, by taking $X = *_{\mathcal{C}}$ we will have proven the existence of a Dirac morphism.

We must show by definition that in the triangle above, the map $f : P \mapsto X$ is such that $res^{\mathcal{C}}_{\mathcal{D}}(f)$ is invertible. Since $res^{\mathcal{C}}_{\mathcal{D}}$ is a triangulated functor we know that

$$res^{\mathcal{C}}_{\mathcal{D}}P \mapsto res^{\mathcal{C}}_{\mathcal{D}}X \mapsto res^{\mathcal{C}}_{\mathcal{D}}N \mapsto res^{\mathcal{C}}_{\mathcal{D}}\Sigma P$$

is an exact triangle. Hence $res^{\mathcal{C}}_{\mathcal{D}}(f)$ is invertible if and only if $res^{\mathcal{C}}_{\mathcal{D}}N = 0$ But this fact is immediate since we know that $Hom_{\mathcal{T}}(A, N) = 0$ for all A in $(\mathcal{CI})$ so that $0 = Hom_{\mathcal{T}}(A, N) = Hom(Ind^{\mathcal{C}}_{\mathcal{D}}W, N) = Hom_{\mathcal{T}}(W, res^{\mathcal{C}}_{\mathcal{D}}N)$ implying that $res^{\mathcal{C}}_{\mathcal{D}}N = 0$

The orthogonality condition $Hom_{\mathcal{T}}(A, N) = 0$ for all A in $(\mathcal{CI})$ can also be obtained in the following way: Consider the functor $A \mapsto Hom(A, X)$ defined in $(\mathcal{CI})$ and where X in $\mathcal{T}$ is fixed. This cohomology functor which takes coproducts into products is defined by what we have already proven, in a well generated localizing category and therefore Brown'srepresentability theorem applies.Hence we have a morphism $f : P \mapsto X$ such that $Hom(A, P) \simeq Hom(A, X)$ where P belongs to $(\mathcal{CI})$. This isomorphism implies by taking $Hom_{\mathcal{T}}$ in the triangle

$$P \mapsto X \mapsto N \mapsto \Sigma P$$

that $Hom_{\mathcal{T}}(A, N) = 0$ for all A in $(\mathcal{CI})$.

REMARK 8: Observe that if Y belongs to $(\mathcal{CI})$ then taking the product in the symmetric monoidal category $Ho(Spt^{Or(G)})$ with another object Z we get that $Y \otimes Z$ is in $(\mathcal{CI})$. This is immediate if we use the following argument:

$Hom_{\mathcal{T}}(Y \otimes Z, W) \simeq Hom_{\mathcal{T}}(Y, Hom_{int}(Z, W)) = Hom_{\mathcal{T}}(IndX, Hom_{int}(Z, W)) \simeq Hom_{\mathcal{T}}(X, Hom_{int}(resZ, resW)) \simeq Hom_{\mathcal{T}}(X \otimes resZ, resW) \simeq Hom_{\mathcal{T}}(Ind(X \otimes resZ), W)$

where Hom_{int} is the internal Hom. Consequently $Y \otimes Z = Ind(X \otimes resZ)$ and we are done.

By remark 8, note that we can go from the Dirac morphism $D : X \mapsto *_{Or(G)}$ to a $(\mathcal{CI})$ approximation of an object K_R with R an associative ring with unit just by taking the product with K_R in the Dirac morphism.

We get $D_R : K_R \otimes X \mapsto K_R$ Therefore if the Dirac morphism D is invertible then so is D_R in which case every K_R is isomorphic to an object in $(\mathcal{CI})$. If that is so, then by theorem 1, the Farrell-Jones conjecture holds.

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